On the Stability of Fluid Flows with Spherical Symmetry

M. S. PLESSET*
California Institute of Technology, Pasadena, California
(Received June 15, 1953)

The conditions for the stability or instability of the interface between two immiscible incompressible fluids in radial motion are deduced. The stability conditions derived by Taylor for the interface of two fluids in plane motion do not apply to spherical flows without significant modifications.

I. INTRODUCTION

THE problem of the stability of a plane interface between two fluids of different densities in accelerated motion has been solved by Taylor,[1] who showed that the interface is stable or unstable according as the acceleration is directed from the heavier to the lighter fluid, or conversely. The corresponding problem for a spherical interface has been discussed recently by Binnie,[2] whose analysis, however, appears to be in error as he has omitted some terms of the same order as those considered in his paper. The problem will be reconsidered here. It may be stated as follows: A fluid of density ρ_1 is contained within a sphere of radius R; a fluid of density ρ_2 occupies the region exterior to this sphere. The fluids will be assumed to be immiscible, incompressible, and nonviscous. If it is assumed that the initial and boundary conditions have spherical symmetry, then the equation of motion for the interface radius R as a function of time is readily determined. The question of present concern is the stability of this spherical interface. The answer to this question has application to the pulsations of underwater explosion bubbles and to the growth or collapse of cavitation bubbles.

The discussion of the stability problem given here will be limited to disturbances of the spherical interface of small amplitude. While the stability question is thereby answered, the results cannot be applied to the determination of the rate of development of interface distortions of significant amplitude.

II. SOLUTION OF THE STABILITY PROBLEM

The origin of a spherical coordinate system is taken at the center of the spherical interface $R(t)$. When the interface is strictly spherical, the velocity potential is

$$\varphi=(R^2\dot{R})/r, \tag{1}$$

where the radial velocity at the point r in the fluid is $-\partial\varphi/\partial r$. This potential, of course, implies a source or sink at the origin according as $\dot{R}$ is positive or negative. The stability of the spherical interface will be established by considering whether a distortion of the interface of small amplitude grows or diminishes. Consider, therefore, a distortion of the interface from R to r_s, where

$$r_s=R+aY_n. \tag{2}$$

Y_n is a spherical harmonic of degree n, and a is a function of time such that

$$|a(t)|\ll R(t).$$

The stability analysis given here will be limited to the first order in a. To this order, the fluid particle velocity at the interface in the radial direction is given by

$$u=\dot{R}+\dot{a}Y_n. \tag{3}$$

Across the interface the normal component of the fluid velocity must be continuous. The difference between the normal component of the fluid velocity at the interface and the radial velocity u of Eq. (3) is of second order in a so that the boundary condition is satisfied by the requirement of continuity of u across the interface.

If one chooses a potential which corresponds to a disturbance which decreases away from the interface in both the inward and outward directions, then one has in place of Eq. (1) the potential

$$\varphi=\varphi_1=[(R^2\dot{R})/r]+b_1r^nY_n, \qquad r<R;$$
$$\varphi=\varphi_2=[(R^2\dot{R})/r]+b_2[Y_n/(r^{n+1})] \qquad r>R.$$

The quantities b_1 and b_2 are determined by the requirement that

$$-\left(\frac{\partial\varphi_1}{\partial r}\right)_{r_s}=-\left(\frac{\partial\varphi_2}{\partial r}\right)_{r_s}=\dot{R}+\dot{a}Y_n,$$

with the result that

$$\varphi_1=\frac{R^2\dot{R}}{r}-\frac{r^n}{nR^{n-1}}Y_n\left[\dot{a}+2a\frac{\dot{R}}{R}\right] \tag{4}$$

and

$$\varphi_2=\frac{R^2\dot{R}}{r}+\frac{R^{n+2}}{(n+1)r^{n+1}}Y_n\left[\dot{a}+2a\frac{\dot{R}}{R}\right], \tag{5}$$

to the first order.

* Consultant to Rand Corporation.
[1] G. I. Taylor, Proc. Roy. Soc. (London) **A201**, 192 (1950).
[2] A. M. Binnie, Proc. Camb. Phil. Soc. **49**, 151 (1953). In the evaluation of the velocity potentials, Binnie has applied the boundary condition of continuity of the radial velocity across the interface to the unperturbed instead of to the perturbed interface. Consequently, an important term is missing from his Eqs. (2) and (3) for the velocity potentials which is of the same order as that retained. In addition, his equation of motion (5) does not include the term $u\partial u/\partial r$ and a term is missing from his evaluation of $\partial u/\partial t$. Finally, the perturbation in the flow is not given by a simple exponential because of the time variation in the unperturbed interface.

One now uses the Bernoulli integral to evaluate the pressure on either side of the interface surface. Thus, if p_1 is the pressure at the interface in region 1 and p_2 is the pressure at the interface in region 2, one has

$$p_1=P_1(t)+\rho_1\left[\left(\frac{\partial\varphi_1}{\partial t}\right)_{r_s}-\tfrac{1}{2}(\mathrm{grad}\,\varphi_1)^2{}_{r_s}\right], \tag{6}$$

$$p_2=P_2(t)+\rho_2\left[\left(\frac{\partial\varphi_2}{\partial t}\right)_{r_s}-\tfrac{1}{2}(\mathrm{grad}\,\varphi_2)^2{}_{r_s}\right]. \tag{7}$$

$P_1(t)$ and $P_2(t)$ are the constants of the spatial integration of the equation of motion which lead to the Bernoulli integral; $P_2(t)$ has the further significance of being the pressure at infinity. The quantities entering in Eqs. (6) and (7) are readily found in the first order to be

$$\begin{aligned}\left(\frac{\partial\varphi_1}{\partial t}\right)_{r_s}=&\frac{1}{R}\frac{d}{dt}(R^2\dot{R})-\frac{aY_n}{R^2}\frac{d}{dt}(R^2\dot{R})\\&-\frac{\ddot{a}}{n}RY_n+\frac{(n-3)}{n}\dot{a}\dot{R}Y_n\\&-\frac{2a}{n}(d^2R/dt^2)Y_n+2a\frac{\dot{R}^2}{R}Y_n;\end{aligned} \tag{8}$$

$$\begin{aligned}\left(\frac{\partial\varphi_2}{\partial t}\right)_{r_s}=&\frac{1}{R}\frac{d}{dt}(R^2\dot{R})-\frac{aY_n}{R^2}\frac{d}{dt}(R^2\dot{R})\\&+\frac{\ddot{a}}{n+1}RY_n+\frac{(n+4)}{(n+1)}\dot{a}\dot{R}Y_n\\&+\frac{2a}{n+1}(d^2R/dt^2)Y_n+2a\frac{\dot{R}^2}{R}Y_n;\end{aligned} \tag{9}$$

$$(\mathrm{grad}\,\varphi_1)^2{}_{r_s}\approx(\mathrm{grad}\,\varphi_2)^2{}_{r_s}\approx u^2\approx\dot{R}^2+2\dot{a}\dot{R}Y_n. \tag{10}$$

It may be noted that while the components of velocity perpendicular to the radial velocity are of first order, their contributions to $(\mathrm{grad}\,\varphi)^2{}_{r_s}$ are of second order and are therefore to be neglected.

The pressures at the interface are connected by the relation

$$p_2=p_1-\sigma(1/R'+1/R''),$$

where R' and R'' are the principle radii of curvature of the interface and σ is the surface tension. To the first order one has[3]

$$\frac{1}{R'}+\frac{1}{R''}=\frac{2}{R}+\frac{(n-1)(n+2)}{R^2}aY_n,$$

so that

$$p_2=p_1-\frac{2\sigma}{R}-\frac{(n-1)(n+2)}{R^2}\sigma aY_n. \tag{11}$$

The terms in this relation between p_2 and p_1 which are independent of Y_n give the equation of motion for the unperturbed interface:

$$R(d^2R/dt^2)+(3/2)\dot{R}^2=(P_1-P_2-2\sigma/R)/(\rho_2-\rho_1). \tag{12}$$

The terms proportional to Y_n in Eq. (11) give the differential equation for a from which stability conditions may be deduced:

$$\ddot{a}+[(3\dot{R})/R]\dot{a}-Aa=0, \tag{13}$$

where

$$A=\frac{[n(n-1)\rho_2-(n+1)(n+2)\rho_1](d^2R/dt^2)-(n-1)n(n+1)(n+2)\sigma/R^2}{R[n\rho_2+(n+1)\rho_1]}. \tag{14}$$

III. DISCUSSION OF THE STABILITY CONDITIONS

The stability of a small-amplitude distortion of a spherical interface is determined by the differential Eq. (13). The nature of the solutions $a(t)$ of this equation follows from the variation of R with time, such variation to be consistent with Eq. (12). It is convenient to make the substitution

$$a=\alpha\exp\left[-\frac{3}{2}\int_{t_0}^{t}\frac{\dot{R}(t')}{R(t')}dt'\right]=\left(\frac{R_0}{R}\right)^{\frac{3}{2}}\alpha, \tag{15}$$

where R_0 is the value of R at some fixed time t_0. Equation (13) then becomes

$$\ddot{\alpha}-G(t)\alpha=0, \tag{16}$$

with

$$\begin{aligned}G(t)&=-\frac{3}{2}\frac{d}{dt}\left(\frac{\dot{R}}{R}\right)+\frac{9}{4}\left(\frac{\dot{R}}{R}\right)^2+A\\&=-\frac{3}{4}\frac{\dot{R}^2}{R^2}+\frac{(d^2R/dt^2)}{R}\left[\frac{3}{2}+\frac{n(n-1)\rho_2-(n+1)(n+2)\rho_1}{n\rho_2+(n+1)\rho_1}\right]\\&\quad-\frac{(n-1)n(n+1)(n+2)\sigma/R^2}{R[n\rho_2+(n+1)\rho_1]}.\end{aligned} \tag{17}$$

It should be noted that the factor $(R_0/R)^{\frac{3}{2}}$ in Eq. (15) is always a stabilizing factor when R is increasing and destabilizing when R is decreasing. Aside from this

[3] H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge, 1932), sixth edition, Sec. 275.

factor, the stability of the deformation is determined by the function $G(t)$ occurring in the differential Eq. (16). As is well known for differential equations of this form,[4] both solutions cannot be bounded for $t>t_0$ when $G(t)>0$; thus one concludes that the deformation is unstable when $G(t)$ is positive. Conversely, the deformation is stable when $G(t)$ is negative.[5]

While the general criterion for instability has been obtained, the complexity of the function G makes simple physical interpretations somewhat obscure so that some special cases may be considered with advantage.

Case 1. $\sigma=0$; $\varrho_2\gg\varrho_1$

The function $G(t)$ simplifies to

$$G(t)=-\frac{3}{4}\frac{\dot{R}^2}{R^2}+\frac{(d^2R/dt^2)}{R}\left[n+\frac{1}{2}\right]. \tag{18}$$

G is certainly positive for $(d^2R/dt^2)\geq 0$ so that one has instability. The surface tension factor which has been neglected is always stabilizing and its effect increases more rapidly with increasing n and decreasing R than the destabilizing terms. As always, the factor $(R_0/R)^{\frac{3}{2}}$ is to be included in an over-all stability determination. It may be remarked that the instability found for $(d^2R/dt^2)>0$ is similar to the instability determined by Taylor for the plane case; there is, however, a possibility for instability in the spherical problem even when $(d^2R/dt^2)<0$. This possibility is evident from Eq. (18). A form of this equation which is useful when ρ_1 is small is obtained by using Eq. (12) which gives for Eq. (18)

$$G(t)=\frac{P_1-P_2}{2\rho R^2}+\frac{n(d^2R/dt^2)}{R}\approx\frac{p_1-P_2}{2\rho R^2}+n\frac{(d^2R/dt^2)}{R}, \tag{19}$$

where p_1 is the pressure at the interface in region 1 and P_2 is the pressure at infinity. It is evident that $G(t)>0$ for $(d^2R/dt^2)<0$ when

$$(p_1-P_2)/\rho>2nR|(d^2R/dt^2)|, \tag{20}$$

where $\rho=\rho_2$. This instability will occur more readily when p_1 is large compared with P_2 and when the values of n are not too great. This type of instability may be the explanation of the deformations on expansion of underwater explosion bubbles[6] and the deformations of pulsating air bubbles.[7]

It may be noted that the above instability conditions apply as well to the case where n is large and where ρ_2 merely exceeds ρ_1.

Case 2. $\sigma=0$; $\varrho_1\gg\varrho_2$

The function $G(t)$ becomes, in this case,

$$G(t)=-\frac{3}{4}\frac{\dot{R}^2}{R^2}-\frac{(d^2R/dt^2)}{R}\left(n+\frac{1}{2}\right). \tag{21}$$

When $(d^2R/dt^2)\leq 0$, $G(t)>0$ so that one has instability. This instability is of the type found by Taylor for the plane case. The effect of the factor $(R_0/R)^{\frac{3}{2}}$ should be kept in mind as well as the obvious stabilizing effect of surface tension. In addition to this Taylor type of instability, there is the possibility of instability for $(d^2R/dt^2)>0$, as is evident from Eq. (21). The condition for this instability may be expressed by the relation

$$R(d^2R/dt^2)<\frac{(3/2)\dot{R}^2}{2n+1},$$

or

$$(d^2R/dt^2)>\frac{P_1-P_2}{2(n+1)\rho_1 R}. \tag{22}$$

---

[4] See, for example, R. Bellman, ONR Report (NAVEXOS P-596), January, 1949.

[5] By stability (or instability) is meant here that there is no exponential increase (or decrease) in the amplitude of the disturbance. There may still be an algebraic increase or decrease in the amplitude such as is shown in Eq. (15).

[6] R. H. Cole, *Underwater Explosions* (Princeton University Press, Princeton, New Jersey, 1948), illustration facing p. 247.

[7] A. T. Ellis, thesis, California Institute of Technology (1953); ONR Report No. 21-12, Hydrodynamics Laboratory, California Institute of Technology.